
The space of solutions to elliptic Monge-Ampère equations in a punctured disc

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Abstract

Let \mathcal{M}_1 denote the space of solutions $z(x, y)$ to an elliptic, real analytic Monge-Ampère equation $\det(D^2z + \mathcal{A}(x, y, z, Dz)) = \varphi(x, y, z, Dz) > 0$ such that: (i) z is defined on a punctured disc around the origin, (ii) z does not extend smoothly across the puncture, and (iii) z, Dz are bounded at the origin. For a large class of such elliptic Monge-Ampère equations (including the case $\mathcal{A} = 0$), we prove that \mathcal{M}_1 is in one-to-one correspondence with a suitable subset of the class \mathcal{M}_2 of regular, real analytic strictly convex Jordan curves in \mathbb{R}^2 . The non-analytic case is also studied.

Among several applications of this classification, we describe the moduli space of bounded isolated singularities of graphs with prescribed analytic positive extrinsic curvature $K_{\text{ext}} = \mathcal{K}(x, y, z, Dz) > 0$ in an arbitrary three-dimensional warped product $M^2 \times_f \mathbb{R}$.

1 Introduction

The study of isolated singularities is a fundamental problem in the theory of nonlinear elliptic PDEs. One of its basic questions is to find conditions under which isolated singularities of a certain PDE are removable. Such conditions typically include that the solution lies in some suitable class of *generalized solutions* (weak solutions, viscosity solutions). However, there exist simple solutions of natural elliptic PDEs that present non-removable isolated singularities, and are therefore uncovered by this analysis.

Consider for instance the radially symmetric function

$$z(r) = \frac{1}{2} \left(r \sqrt{1 + r^2} + \sinh^{-1}(r) \right), \quad r := \sqrt{x^2 + y^2}, \quad (1)$$

whose graph is presented in Figure 1. The function z is a solution to the simplest elliptic Monge-Ampère equation, namely, the *Hessian one equation* $z_{xx}z_{yy} - z_{xy}^2 = 1$. It is real analytic on $\mathbb{R}^2 \setminus \{(0, 0)\}$, and presents at the origin a non-removable isolated singularity around which both the function and its gradient are bounded. This is a prototype for the kind of isolated singularities that will be studied here.

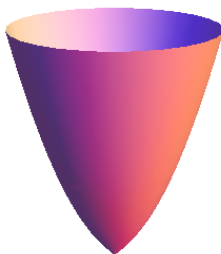


Figure 1: A solution to $\det(D^2z) = 1$ with an isolated singularity.

Consider now the general elliptic equation of Monge-Ampère type in dimension two, which is the following fully nonlinear PDE:

$$\det(D^2z + \mathcal{A}(x, y, z, Dz)) = \varphi(x, y, z, Dz) > 0. \quad (2)$$

Here, Dz , D^2z denote respectively the gradient and the Hessian of z , and $\mathcal{A}(x, y, z, Dz) \in \mathcal{M}_2(\mathbb{R})$ is symmetric. The Monge-Ampère equation (2) can be rewritten as

$$Az_{xx} + 2Bz_{xy} + Cz_{yy} + z_{xx}z_{yy} - z_{xy}^2 = E, \quad (3)$$

where $A = A(x, y, z, z_x, z_y), \dots, E = E(x, y, z, z_x, z_y)$ are defined on an open set $\mathcal{U} \subset \mathbb{R}^5$ and satisfy on \mathcal{U} the ellipticity condition*

$$D := AC - B^2 + E > 0. \quad (4)$$

*It can happen that the coefficients A, \dots, E are actually defined in a larger domain of \mathbb{R}^5 which contains \mathcal{U} (e.g. in all of \mathbb{R}^5), and that the ellipticity condition (4) fails to hold in this larger domain.

We will study solutions z to the elliptic equation (3) around a non-removable isolated singularity, and for simplicity we will assume that this singularity is placed at the origin. All our results can be trivially adapted if the singularity is placed elsewhere.

Convention: From now on we will use the following notations:

- $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < \rho^2\}$, a punctured disc centered at the origin.
- $\mathcal{U} \subset \mathbb{R}^5$ is an open set, and

$$\mathcal{H} := \mathcal{U} \cap \{(x_1, \dots, x_5) \in \mathbb{R}^5 : x_1 = x_2 = x_3 = 0\}.$$

- A, \dots, E are functions in $C^{1,\mu}(\mathcal{U})$, which satisfy in \mathcal{U} the ellipticity condition (4).
- For a function $z \in C^2(\Omega)$, we define

$$\mathfrak{H} := \{(x, y, z(x, y), z_x(x, y), z_y(x, y)) : (x, y) \in \Omega\}. \quad (5)$$

Then, motivated by the example (1), we introduce the following definition.

Definition 1. We say that a solution $z \in C^2(\Omega)$ to (3) for the coefficients A, \dots, E is a singular solution of (3) in Ω if

1. z is not C^1 at the origin.
2. $\overline{\mathfrak{H}}$ is a compact subset of \mathcal{U} .

In the case that the coefficients A, \dots, E are real analytic on \mathcal{U} , the solution z is also real analytic on Ω . Also, it can be easily proved that in the conditions of the definition, $z(x, y)$ extends *continuously* to $(0, 0)$.

We will assume for simplicity from now on that any singular solution to (3) has been continuously extended to the origin by $z(0, 0) = 0$, and that $\mathcal{H} \neq \emptyset$.

Finally, we define the *limit gradient* of z at the origin to be the set $\gamma \subset \mathbb{R}^2$ of points $\xi \in \mathbb{R}^2$ for which there is a sequence $q_n \rightarrow (0, 0)$ in Ω such that $(z_x, z_y)(q_n) \rightarrow \xi$.

Given A, \dots, E real analytic as above, our main results show:

1. For any regular, real analytic, strictly convex Jordan curve $\gamma \subset \mathbb{R}^2$, there is a singular solution $z(x, y)$ to (3) with limit gradient γ . (cf. Theorem 1)
2. Assume that A, \dots, E satisfy condition (\star) below, which is always true for the pure Monge-Ampère equation $\det(D^2 z) = \varphi(x, y, z, Dz) > 0$.

$$\text{The functions } A_p, A_q + 2B_p, C_p + 2B_q \text{ and } C_q \text{ do not depend on } p \text{ and } q \text{ in } \mathcal{U}. \quad (\star)$$

Then (cf. Theorem 2):

- (a) If $z \in C^2(\Omega)$ is a singular solution to (3), then its limit gradient γ is a regular, real analytic, strictly convex Jordan curve.

- (b) Two singular solutions to (3) with the same limit gradient coincide in a neighborhood of the origin.
- 3. If A, \dots, E satisfy condition (\star) , the class \mathcal{M}_1 of singular solutions to (3) is in one-to-one correspondence with the class \mathcal{M}_2 of regular, real analytic, strictly convex Jordan curves $\gamma \subset \mathcal{H} \subset \mathbb{R}^2$ (cf. Theorem 3).

Item 2(a) above holds in the smooth (not necessarily analytic) category. These results generalize many previous theorems on solutions to elliptic Monge-Ampère equations in the presence of isolated singularities, see for instance [ACG, Bey1, Bey2, HeB, GMM, Jor1, Jor2].

We apply the results above to obtain some geometric consequences. These include, in the real analytic category,

- I) The classification of bounded isolated singularities for graphs of prescribed positive extrinsic curvature $K_{\text{ext}} = \mathcal{K}(x, y, z, Dz) > 0$ in an arbitrary warped product three-manifold $M^2 \times_f \mathbb{R}$.
- II) The classification of (bounded or unbounded) isolated singularities for graphs over horospheres in hyperbolic 3-space with prescribed $K_{\text{ext}} > 1$.
- III) The classification of isolated singularities for embedded surfaces in \mathbb{R}^3 of prescribed $K_{\text{ext}} = \mathcal{K}(x, y, z) > 0$.

Again, these classification results rely on establishing a bijection between the corresponding moduli space and a suitable class of real analytic Jordan curves.

The paper is organized as follows. In Section 2 we state accurately our main analytic results (see Theorems 1 and 2) and explain how Theorem 3 follows from them. In Section 3 we give some preliminaries. Theorems 1 and 2 are proved in Section 4 and Section 5, respectively. In Section 6 we provide geometric applications of the classification theorem to the study of isolated singularities in different surface theories. The paper concludes with an Appendix on the geometry of isolated singularities of graphs of positive extrinsic curvature in warped product manifolds $M^2 \times_f \mathbb{R}$.

2 Statement of the main results

In order to state our main results we need to introduce some terminology. Let $z \in C^2(\Omega)$ be a singular solution to (3). It follows then from the ellipticity condition (4) that the expression

$$ds^2 = (z_{xx} + C)dx^2 + 2(z_{xy} - B)dxdy + (z_{yy} + A)dy^2 \quad (6)$$

is a Riemannian metric on Ω . Thus, (Ω, ds^2) admits global conformal parameters, i.e. there exists a diffeomorphism $\Phi : \Omega \rightarrow \Lambda \subset \mathbb{C}$ to a complex domain Λ such that, in coordinates $(u, v) \in \Lambda$, the metric ds^2 is expressed as $ds^2 = \lambda(du^2 + dv^2)$ for some positive function λ on Λ .

In this situation, motivated by [HeB], we introduce the following definition.

Definition 2. A solution z to (3) in Ω satisfies the Heinz-Beyerstedt condition, in short HeB-condition, if A_p , $A_q + 2B_p$, $C_p + 2B_q$ and C_q are Lipschitz continuous in $\overline{\Omega}$ when they are considered as functions of x and y .

It was proved by Heinz and Beyerstedt (cf. [HeB, Lemma 3.3]) that if $z \in C^2(\Omega)$ is a singular solution to (3) which satisfies the HeB-condition, then Λ is conformally equivalent to some annulus \mathbb{A}_R .

Thus, in order to study singular solutions to (3) that satisfy the HeB-condition, we may assume Λ to be a quotient strip $\Gamma_r = \{z \in \mathbb{C} : 0 < \text{Im } z < r\}/(2\pi\mathbb{Z})$, which is obviously conformal to some annulus. From now on, (u, v) will denote the canonical coordinates in this strip.

Let $G = \{(x, y, z(x, y)) : (x, y) \in \Omega\} \subset \mathbb{R}^3$ be the graph of $z(x, y)$. By using the parameters (u, v) , we may parametrize G as a map

$$\psi(u, v) = (x(u, v), y(u, v), z(u, v)) : \Gamma_r \rightarrow G \subset \mathbb{R}^3 \quad (7)$$

such that ψ extends continuously to \mathbb{R} with $\psi(u, 0) = (0, 0, 0)$.

With all of this, our first main result is the following one, which provides a very general existence theorem for real analytic singular solutions to (3).

In what follows, we will denote $\Sigma_r := \{z : 0 < \text{Im } z < r\}$, $\widehat{\Sigma}_r := \{z : -r < \text{Im } z < r\}$, $\Gamma_r := \Sigma_r/(2\pi\mathbb{Z})$ and $\widehat{\Gamma}_r := \widehat{\Sigma}_r/(2\pi\mathbb{Z})$.

Theorem 1. Assume that the coefficients A, \dots, E are real analytic in \mathcal{U} . Let $\gamma(u) = (\alpha(u), \beta(u))$ be a real analytic, 2π -periodic curve such that $(0, 0, 0, \gamma(\mathbb{R})) \subset \mathcal{U}$.

Then, there exists a real analytic map $\psi : \widehat{\Gamma}_r \rightarrow \mathbb{R}^3$ such that:

1. $\psi(u, 0) = (0, 0, 0)$ for every $u \in \mathbb{R}$.
2. There exists a map $(p, q) : \widehat{\Gamma}_r \rightarrow \mathbb{R}^2$ such that $(p, q)(u, 0) = \gamma(u)$ for every $u \in \mathbb{R}$ and $(\psi, p, q)(\Gamma_r) \subset \mathcal{U}$. Moreover, the map $N(u, v) : \widehat{\Gamma}_r \rightarrow \mathbb{S}^2$ defined by

$$N(u, v) = \frac{(-p, -q, 1)}{\sqrt{1 + p^2 + q^2}}(u, v)$$

satisfies that $\langle \psi_u, N \rangle = \langle \psi_v, N \rangle = 0$ in Γ_r .

3. Assume that the map $(x(u, v), y(u, v))$ is an orientation preserving local diffeomorphism at some point $(u_0, v_0) \in \Gamma_r$. Then, the image of ψ around that point is the graph $G \subset \mathbb{R}^3$ of some real analytic solution $z = z(x, y)$ to (3) for the coefficients A, \dots, E .
4. If the curve $\gamma(u)$ is regular, negatively oriented, and strictly convex (i.e. both $-|\gamma'(u)|$ and the curvature of $\gamma(u)$ are strictly negative for every u), then for $r > 0$ small enough, $\psi(\Gamma_r)$ is the graph of a singular solution to (3) for the coefficients A, \dots, E . Moreover, the limit gradient of this solution is the curve $\gamma = \gamma(\mathbb{R})$.

Remark 1. *The first three items of Theorem 1 prove that, if we start from a 2π -periodic, real analytic curve $\gamma(u)$ in \mathbb{R}^2 , we can construct from $\gamma(u)$ a multivalued solution to (3). Here by a multivalued solution we mean a surface such that whenever it is transverse to the vertical direction around one point, it is a local solution to (3) around this point.*

Theorem 2 below provides a converse statement to Theorem 1. In particular, it shows that the limit gradient of a singular solution $z \in C^2(\Omega)$ to (3) that satisfies the HeB-condition is a regular, analytic strictly convex curve that determines $z(x, y)$ uniquely.

In this sense, it is important to observe the following fact (see Lemma 1): *if the coefficients A, \dots, E in (3) satisfy the condition (\star) written in the Introduction, then any singular solution $z \in C^2(\Omega)$ to (3) verifies the HeB-condition.*

We note that condition (\star) holds for the pure Monge-Ampère equation

$$\det(D^2 z) = \varphi(x, y, z, Dz) > 0. \quad (8)$$

Theorem 2. *Assume that the coefficients A, \dots, E are real analytic in \mathcal{U} . Let $z \in C^2(\Omega)$ be a singular solution to (3) which satisfies the HeB-condition. Then:*

1. *The limit gradient of $z(x, y)$ is a regular, strictly convex Jordan curve γ in \mathbb{R}^2 , which is real analytic.*
2. *If (u, v) denote conformal coordinates on Σ_r for the metric ds^2 as explained previously, and $p = z_x, q = z_y$ are viewed as functions of (u, v) , then those functions extend analytically to $\Sigma_r \cup \mathbb{R}$ and $\gamma(u) := (p(u, 0), q(u, 0))$ is an analytic, 2π -periodic, negatively oriented parametrization of γ such that $\gamma'(u) \neq (0, 0)$ for all $u \in \mathbb{R}$.*
3. *The graph G of $z(x, y)$ can be constructed following the procedure described in Theorem 1, in terms of the parameterized limit gradient $\gamma(u)$.*
4. *If $z' \in C^2(\Omega)$ is another singular solution to (3) for the coefficients A, \dots, E , and with the same limit gradient γ , then the graphs of z and z' agree on an open set containing the origin.*

Observe that from Theorem 1, Theorem 2 and Lemma 1 we can fully classify the space of singular solutions to an elliptic Monge-Ampère equation (3) with real analytic coefficients A, \dots, E satisfying the condition (\star) (such as the pure Monge-Ampère equation (8)):

Theorem 3. *Assume that the coefficients A, \dots, E are real analytic and satisfy the condition (\star) . Let us consider the following two classes:*

- $\mathcal{M}_1 = \{ \text{singular solutions to (3) for these coefficients} \}.$
- $\mathcal{M}_2 = \{ \text{regular, analytic, strictly convex Jordan curves } \gamma \text{ in } \mathcal{H} \subset \mathbb{R}^2 \}.$

Then, the map sending each element $z \in C^\omega(\Omega)$ of \mathcal{M}_1 to its limit gradient $\gamma \in \mathbb{R}^2$ gives an explicit one-to-one correspondence between \mathcal{M}_1 and \mathcal{M}_2 .

Remark 1. In Theorem 3, two elements of \mathcal{M}_1 are identified if they agree on an open set around the singularity.

3 Preliminaries

Let $z \in C^2(\Omega)$ be a singular solution to (3), and consider the Riemannian metric ds^2 in (6) associated to z . From now on, we will assume that z as well as the coefficients A, \dots, E of (3) are real analytic (see Subsection 5.1 for a discussion about the non-analytic case). Also, we will be using the standard classical notation $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$.

It is a well known fact that ds^2 admits conformal coordinates $w := u + iv$ such that

$$ds^2 = \frac{\sqrt{D}}{u_x v_y - u_y v_x} |dw|^2. \quad (9)$$

That is, there exists a real analytic diffeomorphism

$$\Phi : \Omega \rightarrow \Lambda := \Phi(\Omega) \subset \mathbb{R}^2, \quad (x, y) \mapsto \Phi(x, y) = (u(x, y), v(x, y)) \quad (10)$$

satisfying

$$x_u y_v - x_v y_u > 0, \quad (11)$$

and the Beltrami system

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{1}{\sqrt{D}} \begin{pmatrix} s - B & -(C + r) \\ A + t & -(s - B) \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}. \quad (12)$$

Here, Λ is a domain in $\mathbb{R}^2 \equiv \mathbb{C}$ which is conformally equivalent to either the punctured disc \mathbb{D}^* or an annulus $\mathbb{A}_R = \{z \in \mathbb{C} : 1 < |z| < R\}$.

Lemma 1. Suppose that the coefficients $A, \dots, E : \mathcal{U} \subset \mathbb{R}^5 \rightarrow \mathbb{R}$ in (3) satisfy the condition:

$$\text{The functions } A_p, A_q + 2B_p, C_p + 2B_q \text{ and } C_q \text{ do not depend on } p \text{ and } q \text{ in } \mathcal{U}. \quad (\star)$$

Then the solution $z(x, y)$ to (3) satisfy the HeB-condition in Definition 2.

Proof. We denote by $F(x, y, z)$ any of the functions in the statement of the condition (\star) . Then, F can be seen as a function \tilde{F} depending on the variables (x, y) using the composition $\tilde{F}(x, y) = (F \circ G)(x, y)$ where $G(x, y) = (x, y, z(x, y))$. On the other hand,

observe that by the regularity of \tilde{F} away from the origin, to prove that the HeB-condition holds we only need to show that for any $(x, y) \in \Omega$:

$$|\tilde{F}(x, y) - \tilde{F}(0, 0)| \leq c|(x, y)|, \quad \text{for some } c > 0. \quad (13)$$

By the hypothesis on the coefficients A, \dots, E , the partial derivatives of the singular solution p and q are bounded in Ω and so $z \in C^{0,1}(\overline{\Omega})$ (see [GiTr, pag. 154]). Then, as F is analytic in \mathcal{U} we deduce:

$$|F(x, y, z) - F(0, 0, 0)| \leq c_1|(x, y, z)| \leq c_1(|(x, y)| + |z|) \leq c_1(|(x, y)| + c_2|(x, y)|),$$

for certain constants $c_1, c_2 > 0$. That is, (13) holds. \square

Note that the analyticity of the coefficients A, \dots, E is a sufficient condition in the previous proof but it is not necessary. If we suppose just that the coefficients are of class $C^{1,1}(\mathcal{U})$, the function F as above is Lipschitz continuous in \mathcal{U} and we can proceed in the same way.

The HeB-condition is used in [HeB, Lemma 3.3] and provides the following classification result:

Lemma 2. *Suppose that the HeB-condition is satisfied. Then, a solution to (3) is singular if and only if Λ is conformally equivalent to some annulus \mathbb{A}_R .*

Thus, in order to study singular solutions to (3) when the coefficients satisfy the HeB-condition we may consider $\Lambda = \mathbb{A}_R$. If we denote $\Sigma_r := \{z : 0 < \text{Im}(z) < r\}$, then \mathbb{A}_R is conformally equivalent to $\Gamma_r := \Sigma_r / (2\pi\mathbb{Z})$ for $r = \log R$. So, composing with this conformal equivalence we will suppose that the map Φ in (10) is a diffeomorphism from Ω into Γ_r ; in particular, Φ is 2π -periodic and (u, v) will denote the canonical coordinates of the strip Σ_r .

From now on, we will consider all the functions depending on the parameters (u, v) via $(x, y) = \Phi^{-1}(u, v)$. For simplicity, we keep the same notation.

From system (12) (see for example [Bey1]) we have the following equations:

$$\begin{aligned} p_u &= \sqrt{D}y_v + By_u - Cx_u, \\ p_v &= -\sqrt{D}y_u + By_v - Cx_v, \\ q_u &= -\sqrt{D}x_v + Bx_u - Ay_u, \\ q_v &= \sqrt{D}x_u + Bx_v - Ay_v. \end{aligned} \quad (14)$$

Moreover we have that

$$\begin{aligned} z_v &= px_v + qy_v \\ &= -\frac{p}{\sqrt{D}}(q_u - Bx_u + Ay_u) + \frac{q}{\sqrt{D}}(p_u - By_u + Cx_u) \\ &= \frac{1}{\sqrt{D}}(x_u(Bp + Cq) - y_u(Ap + Bq) + qp_u - pq_u). \end{aligned}$$

So, the following system holds:

$$\begin{pmatrix} x \\ y \\ z \\ p \\ q \end{pmatrix}_v = \widetilde{M} \begin{pmatrix} x \\ y \\ z \\ p \\ q \end{pmatrix}_u, \quad \widetilde{M} = \frac{1}{\sqrt{D}} \begin{pmatrix} B & -A & 0 & 0 & -1 \\ C & -B & 0 & 1 & 0 \\ Bp + Cq & -Ap - Bq & 0 & q & -p \\ 0 & -E & 0 & B & C \\ E & 0 & 0 & -A & -B \end{pmatrix}. \quad (15)$$

We conclude these preliminaries with a useful lemma.

Lemma 3. *Let $z \in C^2(\Omega)$ be a singular solution to (3), where Ω is an open domain of \mathbb{R}^2 (not necessarily a punctured disc). Then, for sufficiently large constants $a, c > 0$, the function*

$$z^*(x, y) = z(x, y) + \frac{a}{2}x^2 + \frac{c}{2}y^2 \quad (16)$$

satisfies that its graph $\{(x, y, z^(x, y)) : (x, y) \in \Omega\}$ is a locally convex surface in \mathbb{R}^3 .*

Proof. As the coefficients A, \dots, E are bounded on \mathfrak{h} , we can find constants $a, c > 0$ such that

$$c - C > 0, \quad a - A > 0, \quad (c - C)(a - A) - B^2 > 0,$$

i.e. the matrix

$$\mathcal{N} = \begin{pmatrix} c - C & B \\ B & a - A \end{pmatrix} \quad (17)$$

is positive definite. Now, we may use the fact that ds^2 in (6) is positive definite to conclude that the symmetric bilinear form

$$ds^2 + (dx, dy)\mathcal{N}(dx, dy)^T = (dx, dy) \begin{pmatrix} r + c & s \\ s & t + a \end{pmatrix} (dx, dy)^T$$

is also positive definite, that is, it is a Riemannian metric on Ω .

On the other hand, a straightforward computation shows that the matrix of the second fundamental form of the graph of $z^*(x, y)$ in (16) is given by

$$\Pi^* = \frac{1}{\sqrt{1 + (p + cx)^2 + (q + ay)^2}} \begin{pmatrix} r + c & s \\ s & t + a \end{pmatrix},$$

which we have just proved is positive definite on Ω . In particular, the graph of $z^*(x, y)$ has positive curvature at every point, which proves the assertion. \square

4 Existence: Proof of Theorem 1

In this section we prove Theorem 1. So, let $\gamma(u) = (\alpha(u), \beta(u))$ be a real analytic, 2π -periodic curve in \mathbb{R}^2 , and assume that A, \dots, E are real analytic functions on an open set $\mathcal{U} \subset \mathbb{R}^5$ that contains $(0, 0, 0, \gamma(\mathbb{R}))$, and that satisfy the ellipticity condition (4). We wish to construct a singular solution to (3) on a punctured disc $\Omega \subset \mathbb{R}^2$ for

these coefficients, so that the limit gradient of the solution at the singularity is exactly $\gamma(\mathbb{R})$.

To start, let us consider the 2π -periodic initial data $(0, 0, 0, \alpha(u), \beta(u))$ along the axis $v = 0$ in the (u, v) -plane for the system (15). By the Cauchy-Kowalevsky theorem, there exists a unique real analytic solution (x, y, z, p, q) to (15), defined on a neighborhood $\widehat{\Sigma}_r = \{(u, v) : -r < v < r\}$ of the axis $v = 0$, such that

$$(x, y, z, p, q)(u, 0) = (0, 0, 0, \alpha(u), \beta(u)).$$

Observe that $\Psi := (x, y, z, p, q) : \widehat{\Sigma}_r \rightarrow \mathbb{R}^5$ is 2π -periodic with respect to u , i.e. it is well defined on the quotient $\widehat{\Gamma}_r := \widehat{\Sigma}_r / (2\pi\mathbb{Z})$.

It is clear from the way we obtained system (15) that p and q satisfy (14), and also that $z_v = px_v + qy_v$. Besides, a computation from (14) proves the relation

$$p_v x_u + q_v y_u = p_u x_v + q_u y_v, \quad (18)$$

which is exactly the integrability condition needed for the existence of some smooth function z_0 on Σ_r , unique up to an additive constant, such that

$$(z_0)_u = px_u + qy_u, \quad (z_0)_v = px_v + qy_v.$$

Note that $(z_0)_v = z_v$ and so $z(u, v) = z_0(u, v) + f(u)$ for a certain real valued function f . Also, observe that the initial conditions that the solution (x, y, z, p, q) to (15) satisfies imply that $z(u, 0) \equiv 0$ and $(z_0)_u(u, 0) \equiv 0$. Thus, $f(u)$ must be constant, and as z_0 was defined up to additive constants we may assume that $z(u, v) = z_0(u, v)$. In particular, it holds

$$z_u = px_u + qy_u, \quad z_v = px_v + qy_v. \quad (19)$$

Defining now

$$\psi(u, v) := (x(u, v), y(u, v), z(u, v)) : \widehat{\Gamma}_r \rightarrow \mathbb{R}^3 \quad (20)$$

and

$$N(u, v) := \frac{(-p(u, v), -q(u, v), 1)}{\sqrt{1 + p(u, v)^2 + q(u, v)^2}} : \widehat{\Gamma}_r \rightarrow \mathbb{S}^2 \quad (21)$$

we see that the first two items of Theorem 1 hold.

To prove item 3, suppose now that the map $(x(u, v), y(u, v))$ is an orientation pre-serving local diffeomorphism at some point $(u_0, v_0) \in \Sigma_r$, i.e. the condition

$$\omega := x_u y_v - x_v y_u > 0 \quad (22)$$

holds at this point. By the Inverse Function Theorem we may write z, p, q in terms of the coordinates x, y . Thus, around (u_0, v_0) the image of the map $\psi(u, v)$ is the graph G in \mathbb{R}^3 of the real analytic function $z = z(x, y)$, and from formula (19) the relations $z_x = p$ and $z_y = q$ hold. We prove next that $z(x, y)$ is a solution to (3) for the coefficients A, \dots, E we started with.

If we recall the notation $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$, then using (14) and working in terms of the (u, v) coordinates we obtain

$$\sqrt{D}y_v = p_u - By_u + Cx_u = (C + r)x_u - (B - s)y_u,$$

and working similarly,

$$\begin{aligned}\sqrt{D}y_u &= -(C + r)x_v + (B - s)y_v, \\ \sqrt{D}x_v &= (B - s)x_u - (A + t)y_u, \\ \sqrt{D}x_u &= -(B - s)x_v + (A + t)y_v.\end{aligned}$$

After the change of coordinates $(u, v) \mapsto (x, y)$, these expressions yield

$$\begin{aligned}u_x &= \frac{(C + r)v_y + (B - s)v_x}{\sqrt{D}}, & v_x &= \frac{-(C + r)u_y - (B - s)u_x}{\sqrt{D}}, \\ u_y &= \frac{-(B - s)v_y - (A + t)v_x}{\sqrt{D}}, & v_y &= \frac{(B - s)u_y + (A + t)u_x}{\sqrt{D}}.\end{aligned}\tag{23}$$

We deduce then from the second and fourth equation in (23) that the system

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \mathfrak{M}_1 \begin{pmatrix} u_x \\ u_y \end{pmatrix}\tag{24}$$

holds, where

$$\mathfrak{M}_1 = \frac{1}{\sqrt{D}} \begin{pmatrix} -(B - s) & -(C + r) \\ A + t & B - s \end{pmatrix}.$$

Similarly, from the first and the third equation in (23) we get

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \mathfrak{M}_2 \begin{pmatrix} v_x \\ v_y \end{pmatrix},\tag{25}$$

where

$$\mathfrak{M}_2 = \frac{1}{\sqrt{D}} \begin{pmatrix} B - s & (C + r) \\ -(A + t) & -(B - s) \end{pmatrix}.$$

Clearly, \mathfrak{M}_1 is proportional to \mathfrak{M}_2^{-1} , i.e. $\mathfrak{M}_1\mathfrak{M}_2 = \lambda(x, y)\text{Id}$ for some function λ . Hence, from (24) and (25) we obtain $\lambda = 1$, i.e. $\mathfrak{M}_1\mathfrak{M}_2 = \text{Id}$, and so

$$(A + t)(C + r) - (B - s)^2 = D.$$

That is, $z(x, y)$ is a solution to (3), as we wanted to show. This completes the proof of item 3.

To prove item 4, assume that $\gamma(u) = (\alpha(u), \beta(u))$ is also regular, negatively oriented and strictly convex, i.e. $\alpha''(u)\beta'(u) - \beta''(u)\alpha'(u) > 0$ for every u . If we let ω be the function in (22), then $\omega(u, 0) = 0$ for every u , and a computation from (14) yields

$$\begin{aligned}
\omega_v(u, 0) &= (x_{uv}y_v - x_vy_{uv})(u, 0) \\
&= \left(\left(-\frac{\beta'}{\sqrt{D}} \right)_u \frac{\alpha'}{\sqrt{D}} + \frac{\beta'}{\sqrt{D}} \left(\frac{\alpha'}{\sqrt{D}} \right)_u \right) (u, 0) \\
&= \frac{-1}{D^{3/2}} \left((\beta''\sqrt{D} - \beta'(\sqrt{D})_u)\alpha' - \beta'(\alpha''\sqrt{D} - \alpha'(\sqrt{D})_u) \right) (u, 0) \\
&= \frac{1}{D} (\beta'(u)\alpha''(u) - \beta''(u)\alpha'(u)) > 0.
\end{aligned} \tag{26}$$

Consequently, since ω is 2π -periodic, there is some $r > 0$ such that $\omega > 0$ on Γ_r . In particular, the map $\psi(u, v) : \widehat{\Gamma}_r \rightarrow \mathbb{R}^3$ given by (20) satisfies:

1. The projection $(x(u, v), y(u, v)) : \Gamma_r \rightarrow \mathbb{R}^2$ is a local diffeomorphism.
2. $\psi(u, 0) = 0$ for every $u \in \mathbb{R}$.
3. The upwards-pointing unit normal $N : \Gamma_r \rightarrow \mathbb{S}_+^2$ of ψ restricted to Γ_r extends analytically to $\widehat{\Gamma}_r$, and (21) holds.

We need to prove now that for $r > 0$ small enough, $\psi(\Gamma_r)$ is a graph of a function $z = z(x, y)$ over a punctured disc $\Omega \subset \mathbb{R}^2$.

First, we observe that a straightforward modification of the proof of Lemma 3 ensures that there exist sufficiently large constants $a, c > 0$ such that the map $\psi^* : \widehat{\Gamma}_r \rightarrow \mathbb{R}^3$ given by

$$\psi^*(u, v) = \left(x(u, v), y(u, v), z(u, v) + \frac{a}{2}x(u, v)^2 + \frac{c}{2}y(u, v)^2 \right)$$

is a regular, strictly convex surface in \mathbb{R}^3 when restricted to Γ_r . Also, $\psi^*(u, 0) = 0$ for every u , and the projection of $\psi^*|_{\Gamma_r}$ to \mathbb{R}^2 is a local diffeomorphism. Moreover, a direct computation shows that the unit normal of ψ^* in Γ_r is

$$N^*(u, v) = \frac{1}{\sqrt{1 + (p + cx)^2 + (q + ay)^2}}(-p - cx, -q - ay, 1),$$

where x, y, p, q are evaluated at (u, v) . We remark that

$$N^*(u, 0) = N(u, 0) = \frac{(-\alpha(u), -\beta(u), 1)}{\sqrt{1 + \alpha(u)^2 + \beta(u)^2}}, \tag{27}$$

which is a regular, convex Jordan curve in the upper hemisphere of \mathbb{S}^2 .

Consider now the *Legendre transform* of $\psi^*(u, v)$, given by (see [LSZ, pag. 89])

$$\mathcal{L}(u, v) = \left(-\frac{N_1^*}{N_3^*}, -\frac{N_2^*}{N_3^*}, -x\frac{N_1^*}{N_3^*} - y\frac{N_2^*}{N_3^*} - z^* \right) : \Gamma_r \rightarrow \mathbb{R}^3,$$

where we are denoting $N^* = (N_1^*, N_2^*, N_3^*)$ and $\psi^* = (x, y, z^*)$. It is well known that, since ψ^* is a regular, locally convex surface in \mathbb{R}^3 whose projection to the (x, y) -plane is a local diffeomorphism, then so is \mathcal{L} . Its upwards-pointing unit normal is

$$\mathcal{N}_{\mathcal{L}} = \frac{(-x, -y, 1)}{\sqrt{1 + x^2 + y^2}} : \Gamma_r \rightarrow \mathbb{S}_+^2, \quad (28)$$

where x, y are evaluated at (u, v) . In particular $\mathcal{L}(u, 0) = (\alpha(u), \beta(u), 0)$ is a regular, strictly convex Jordan curve in \mathbb{R}^2 and $\mathcal{N}_{\mathcal{L}}(u, 0) = (0, 0, 1)$. Therefore, for $r > 0$ small enough, $\mathcal{L}(\Gamma_r)$ lies on the upper half-space of \mathbb{R}^3 . Moreover, the intersection of $\mathcal{L}(\Gamma_r)$ with each plane $z = \varepsilon$ for ε small enough is a regular convex Jordan curve in that plane. In particular, the piece of the surface $\mathcal{L}(\Gamma_r)$ lying between two of those parallel planes associated to $0 < \varepsilon_1 < \varepsilon_2$ is strictly convex, and bounded by two regular convex Jordan curves, one on each plane. In these conditions, the unit normal of this piece of \mathcal{L} defines a global diffeomorphism onto some annular domain of \mathbb{S}_+^2 . Letting $\varepsilon_2 \rightarrow 0$ we conclude that there exists some $r > 0$ small enough such that the unit normal (28) to \mathcal{L} restricted to Γ_r is a diffeomorphism onto a domain of \mathbb{S}^2 . But now, in the view of the expression (28), this means that the map $(x(u, v), y(u, v))$ restricted to this domain Γ_r is a global diffeomorphism onto its image. Thus, both $\psi(\Gamma_r)$ and $\psi^*(\Gamma_r)$ are graphs of functions $z(x, y)$ and $z^*(x, y)$ over a punctured disc $\Omega \subset \mathbb{R}^2$.

Observe that by item 3, the function $z(x, y)$ is a singular solution to (3). Moreover, it is clear from the construction process we have followed that its limit gradient at the singularity is the curve γ we started with. This concludes the proof of item 4 and Theorem 1.

5 Uniqueness: Proof of Theorem 2

Along this section, $z(x, y)$ will denote a singular solution to (3) on a punctured disc Ω , with real analytic coefficients A, \dots, E . Moreover, we will suppose that HeB-condition holds which by Lemma 2 shows that the conformal structure induced by the metric ds^2 in (9) is that of an annulus. That is, we can parametrize the graph $G \subset \mathbb{R}^3$ of z as (7) for some $r > 0$, so that (u, v) are conformal parameters in Γ_r for the metric ds^2 in (6). Observe that ψ extends to \mathbb{R} as $\psi(u, 0) = (0, 0, 0)$ for all u .

If we also parametrize $p = z_x, q = z_y$ in terms of u, v , then

$$\mathbf{z}(u, v) := (x(u, v), y(u, v), z(u, v), p(u, v), q(u, v)) : \Gamma_r \longrightarrow \mathbb{R}^5$$

is a solution to system (15). The following proposition provides a boundary regularity result for $\mathbf{z}(u, v)$:

Proposition 1. *In the above conditions, $\mathbf{z}(u, v)$ extends as a real analytic map to $\Gamma_r \cup \mathbb{R}$.*

Proof. The first part of the proof follows a bootstrapping method. Consider an arbitrary point of \mathbb{R} , which we will suppose without loss of generality to be the origin. Also, consider for $0 < \delta < r$ the domain $\mathbb{D}^+ = \{(u, v) : 0 < u^2 + v^2 < \delta^2\} \cap \Gamma_r$.

From (14) it follows that (cf. [HeB])

$$\begin{aligned}\Delta x &= h_1(x_u^2 + x_v^2) + h_2(x_u y_u + x_v y_v) + h_3(y_u^2 + y_v^2) + h_4(x_u y_v - x_v y_u) \\ \Delta y &= \tilde{h}_1(x_u^2 + x_v^2) + \tilde{h}_2(x_u y_u + x_v y_v) + \tilde{h}_3(y_u^2 + y_v^2) + \tilde{h}_4(x_u y_v - x_v y_u)\end{aligned}\quad (29)$$

where the coefficients $h_1 = h_1(x, y, z, p, q), \dots, \tilde{h}_4 = \tilde{h}_4(x, y, z, p, q)$ are

$$\begin{aligned}h_1 &= B_q - \frac{1}{2D}(D_x + D_z p - D_p C + D_q B), \\ h_2 &= -A_q - B_p - \frac{1}{2D}(D_y + D_z q + D_p B - D_q A), \\ h_3 &= A_p, \\ h_4 &= \frac{1}{\sqrt{D}}(A_x + B_y + A_z p + B_z q - A_p C + (A_q + B_p)B - B_q A - \frac{1}{2}D_p), \\ \tilde{h}_1 &= C_q, \\ \tilde{h}_2 &= -B_q - C_p - \frac{1}{2D}(D_x + D_z p - D_p C + D_q B), \\ \tilde{h}_3 &= B_p - \frac{1}{2D}(D_y + D_z q + D_p B - D_q A), \\ \tilde{h}_4 &= \frac{1}{\sqrt{D}}(C_y + B_x + C_z q + B_z p - B_p C + (B_q + C_p)B - C_q A - \frac{1}{2}D_q),\end{aligned}$$

all of them evaluated at $\mathbf{z}(u, v)$. In particular we have that

$$\tilde{h}_1 = C_q, \quad h_1 - \tilde{h}_2 = C_p + 2B_q, \quad h_2 - \tilde{h}_3 = -A_q - 2B_p, \quad h_3 = A_p. \quad (30)$$

On the other hand, observe that the inequalities

$$(x_u - y_v)^2 + (x_v + y_u)^2 \geq 0, \quad (x_u - y_u)^2 + (x_v - y_v)^2 \geq 0$$

lead, respectively, to $x_u y_v - x_v y_u \leq \frac{1}{2}(|\nabla x|^2 + |\nabla y|^2)$ and $x_u y_u + x_v y_v \leq \frac{1}{2}(|\nabla x|^2 + |\nabla y|^2)$.

Hence, if we denote $Y = (x, y) : \mathbb{D}^+ \longrightarrow \Omega$, formula (29) and the fact that h_1, \dots, \tilde{h}_4 are bounded yield

$$|\Delta Y| \leq c(|\nabla x|^2 + |\nabla y|^2) \quad (31)$$

for a certain constant $c > 0$.

Observe that $Y \in C^2(\mathbb{D}^+) \cap C^0(\overline{\mathbb{D}^+})$ with $Y(u, 0) = (0, 0)$ for all u . Hence, we can apply Heinz's Theorem in [He] to deduce that $Y \in C^{1,\alpha}(\overline{\mathbb{D}_\varepsilon^+})$ for all $\alpha \in (0, 1)$, where $\mathbb{D}_\varepsilon^+ = \mathbb{D}^+ \cap B(0, \varepsilon)$ for a certain $0 < \varepsilon < \delta$.

Now, the right hand side terms in (14) are bounded in $\overline{\mathbb{D}_\varepsilon^+}$ and so $p, q \in W^{1,\infty}(\overline{\mathbb{D}_\varepsilon^+})$. Hence $p, q \in C^{0,1}(\overline{\mathbb{D}_\varepsilon^+})$ (cf. [GiTr, pag. 154]). In particular, p and q are Hölder continuous of any order in $\overline{\mathbb{D}_\varepsilon^+}$.

Taking into account (19), we obtain $z \in C^{1,\alpha}(\overline{\mathbb{D}_\varepsilon^+}) \forall \alpha \in (0, 1)$. Then, the right hand side functions in (14) are Hölder continuous of any order in $\overline{\mathbb{D}_\varepsilon^+}$. That is, $p, q \in C^{1,\alpha}(\overline{\mathbb{D}_\varepsilon^+}) \forall \alpha \in (0, 1)$.

With this, we have from (29) that ΔY is Hölder continuous in $\overline{\mathbb{D}_\varepsilon^+}$. Then, a standard potential analysis argument (cf. [GiTr, Lemma 4.10]) ensures that $x, y \in C^{2,\alpha}(\overline{\mathbb{D}_{\varepsilon/2}^+})$. Again, by formula (14) we have that $p, q \in C^{2,\alpha}(\overline{\mathbb{D}_{\varepsilon/2}^+})$ and so, from (19) that $z \in C^{2,\alpha}(\overline{\mathbb{D}_{\varepsilon/2}^+})$.

At this point, we may apply the same argument to Y_u and Y_v , in order to obtain that $x, y, z, p, q \in C^{3,\alpha}(\overline{\mathbb{D}_{\varepsilon/4}^+})$. A recursive process leads to the fact that $\mathbf{z} = (x, y, z, p, q)$

is C^∞ at the origin. As we can do the same argument for all points of \mathbb{R} and not just the origin, we conclude that $\mathbf{z}(u, v) \in C^\infty(\Gamma_r \cup \mathbb{R})$.

Now, a computation in the same spirit of formula (29) shows that the Laplacians of z, p, q are given by:

$$\begin{aligned}\Delta p &= (\sqrt{D \circ \mathbf{z}})_u y_v - (\sqrt{D \circ \mathbf{z}})_v y_u + (B \circ \mathbf{z})_u y_u + (B \circ \mathbf{z})_v y_v \\ &\quad + (B \circ \mathbf{z}) \Delta y - (C \circ \mathbf{z}) \Delta x - (C \circ \mathbf{z})_u x_u - (C \circ \mathbf{z})_v x_v, \\ \Delta q &= -(\sqrt{D \circ \mathbf{z}})_u x_v + (\sqrt{D \circ \mathbf{z}})_v x_u + (B \circ \mathbf{z})_u x_u + (B \circ \mathbf{z})_v x_v \\ &\quad + (B \circ \mathbf{z}) \Delta x - (A \circ \mathbf{z}) \Delta y - (A \circ \mathbf{z})_u y_u - (A \circ \mathbf{z})_v y_v, \\ \Delta z &= p_u x_u + p_v x_v + q_u y_u + q_v y_v + p \Delta x + q \Delta y,\end{aligned}\tag{32}$$

where all quantities are evaluated at $(u, v) \in \Gamma_r$. Therefore, as A, B, C, E are analytic (note that the analyticity of the coefficients had not been used in this section up to now), $\mathbf{z}(u, v)$ satisfies

$$\Delta \mathbf{z} = h(\mathbf{z}, \mathbf{z}_u, \mathbf{z}_v) \tag{33}$$

where $h : \mathcal{O} \subset \mathbb{R}^{15} \rightarrow \mathbb{R}^5$ is a real analytic function on an open set \mathcal{O} of \mathbb{R}^{15} containing the closure of the bounded set $\{(\mathbf{z}, \mathbf{z}_u, \mathbf{z}_v)(u, v) : (u, v) \in \Gamma_r\}$. Moreover, if we write

$$\mathbf{z}(u, v) = (\psi(u, v), \phi(u, v)) : \Gamma_r \rightarrow \mathbb{R}^3 \times \mathbb{R}^2 \equiv \mathbb{R}^5$$

where $\psi(u, v)$ is given by (7) and $\phi(u, v) = (p(u, v), q(u, v))$, then we see that $\mathbf{z}(u, v)$ is a solution to (33) that meets the mixed initial conditions

$$\begin{cases} \psi(u, 0) = (0, 0, 0), \\ \phi_v(u, 0) = \begin{pmatrix} -C(0, 0, 0, \phi(u, 0)) & B(0, 0, 0, \phi(u, 0)) & 0 \\ B(0, 0, 0, \phi(u, 0)) & -A(0, 0, 0, \phi(u, 0)) & 0 \end{pmatrix} \psi_v(u, 0)^T. \end{cases}$$

As $\mathbf{z} \in C^\infty(\Gamma_r \cup \mathbb{R})$, we are in the conditions to apply Theorem 3 in [Mu2] to \mathbf{z} around every point in \mathbb{R} . Thus, we conclude that \mathbf{z} is real analytic in $\Gamma_r \cup \mathbb{R}$, which concludes the proof of Proposition 1. \square

It follows from Proposition 1 that the functions $p(u, v)$ and $q(u, v)$ extend analytically to $\Gamma_r \cup \mathbb{R}$, so that $(\alpha(u), \beta(u)) := (p(u, 0), q(u, 0))$ is a real analytic, 2π -periodic map. Let now $\gamma \subset \mathbb{R}^2$ denote the limit gradient of $z(x, y)$. Then, clearly $\gamma = \{(\alpha(u), \beta(u)) : u \in \mathbb{R}\}$, and so we get that γ is a closed curve in \mathbb{R}^2 , possibly with singularities, that can be parameterized as a 2π -periodic function as $\gamma(u) = (\alpha(u), \beta(u))$ in terms of the conformal parameters (u, v) associated to the solution $z(x, y)$.

In Theorem 1 we proved that a sufficient condition for the possibly multivalued solution to (3) constructed there in terms of $\gamma(u)$ to be actually a singular solution to (3) on Ω is that $\gamma(u)$ is regular, negatively oriented and strictly convex. We show next that under the hypothesis that the HeB-condition holds these conditions on $\gamma(u)$ are also necessary. The following proposition completes the proof of items 1 and 2 of Theorem 2.

Proposition 2. *Let $\gamma \subset \mathbb{R}^2$ be the limit gradient of $z(x, y)$, and let $\gamma(u) : \mathbb{R}/(2\pi\mathbb{Z}) \rightarrow \mathbb{R}^2$ be its parametrization in terms of the conformal coordinates $(u, v) \in \Gamma_r \cup \mathbb{R}$. Then $\langle \gamma''(u), J\gamma'(u) \rangle < 0$ for all $u \in \mathbb{R}$, where J denotes the rotation of angle $\pi/2$. That is, $\gamma(u)$ is a regular analytic negatively oriented Jordan curve which, in particular, is strictly convex and bounds a convex set in \mathbb{R}^2 .*

Proof. That $\gamma(u)$ is real analytic was proved in Proposition 1.

Consider next the analytic map ω defined on $\Gamma_r \cup \mathbb{R}$ by (22), which vanishes on \mathbb{R} and is positive on Γ_r .

We can use (29) to compute the Laplacian of ω in (22) at points in the real axis. We get

$$\begin{aligned}
\Delta\omega(u, 0) = \omega_{vv}(u, 0) &= (x_{uvv}y_v - x_v y_{uvv} + 2(x_{uv}y_{vv} - x_{vv}y_{uv}))(u, 0) \\
&= [(h_1)_u x_v^2 y_v + 2h_1 x_v y_v x_{vu} \\
&\quad + (h_2)_u x_v y_v^2 + h_2 x_{vu} y_v^2 + h_2 x_v y_v y_{vu} \\
&\quad + (h_3)_u y_v^3 + 2h_3 y_v^2 y_{vu} \\
&\quad - (\tilde{h}_1)_u x_v^3 - 2\tilde{h}_1 x_v^2 x_{vu} \\
&\quad - (\tilde{h}_2)_u x_v^2 y_v - \tilde{h}_2 x_{vu} y_v x_v - \tilde{h}_2 x_v^2 y_{vu} \\
&\quad - (\tilde{h}_3)_u y_v^2 x_v - 2\tilde{h}_3 y_v x_v y_{vu} \\
&\quad + 2x_{uv}(\tilde{h}_1 x_v^2 + \tilde{h}_2 x_v y_v + \tilde{h}_3 y_v^2) \\
&\quad - 2y_{uv}(h_1 x_v^2 + h_2 x_v y_v + h_3 y_v^2)](u, 0) \\
&= [(h_1 - \tilde{h}_2)_u x_v^2 y_v + (h_2 - \tilde{h}_3)_u y_v^2 x_v + (h_3)_u y_v^3 - (\tilde{h}_1)_u x_v^3 \\
&\quad + (x_v(\tilde{h}_2 + 2h_1) + y_v(h_2 + 2\tilde{h}_3))\omega_v](u, 0).
\end{aligned} \tag{34}$$

Observe that since HeB-condition holds, the functions in (30) are constant along the real axis. With this, and using that x_v and y_v are bounded from above along the real axis by (14), we deduce from (34) that there exists a constant $c > 0$ such that

$$\Delta\omega \leq c \omega_v \tag{35}$$

for all $(u, 0) \in \mathbb{R}^2$. Therefore, (35) holds in some Γ_ρ with $r > \rho > 0$. As $\omega(u, 0) = 0$ and $\omega(u, v) > 0$ in Γ_r , Hopf's Lemma (see [GiTr, Lemma 3.4]) implies that $\omega_v(u, 0) > 0$. Using now (26) we get that $\langle \gamma''(u), J\gamma'(u) \rangle < 0$ at every point. Hence, $\gamma'(u) \neq 0$ for all u and the curve is negatively oriented and strictly convex. Thus, the proof of Proposition 2 is finished. \square

Now we are in conditions to finish the proof of Theorem 2. Observe that the map $\mathbf{z}(u, v)$ can be recovered in terms of a real analytic, 2π -periodic curve $\gamma(u) = (\alpha(u), \beta(u))$ as the unique solution to the Cauchy problem for the system (15) with the initial condition

$$\mathbf{z}(u, 0) = (0, 0, 0, \alpha(u), \beta(u)). \tag{36}$$

In other words, any singular solution to (3) on a punctured disc Ω , whenever the HeB-condition is satisfied, can be recovered through the process described in Theorem 1 to construct solutions to (3) with an isolated singularity at the origin. This shows that item 3 in Theorem 2 holds.

Also, item 4 is a direct consequence of the previous facts and the uniqueness of the solution to the Cauchy problem for system (15).

Remark 2. *Observe that the parameters $(u, v) \in \Gamma_r$ associated to the solution z of (3) are defined up to 2π -periodic conformal changes of Γ_r that simply yield a reparametrization of the limit gradient γ . Hence, the graph constructed in Section 4 is uniquely determined by the initial data γ , independently of its parametrization.*

This finishes the proof of Theorem 2.

5.1 The non-analytic case

In the case that the coefficients A, \dots, E in (3) are *not* real analytic, the proof of Theorem 1 does not work since it uses fundamentally the Cauchy-Kowalevsky theorem. Thus, we cannot construct singular solutions to (3) with a prescribed limit gradient at the singularity.

However, our method still provides information about the asymptotic behavior of singular solutions to (3) in the non-analytic setting. Indeed, we have.

Corollary 1. *Assume that the coefficients A, \dots, E in (3) are of class $C^{1,1}$ in $\mathcal{U} \subset \mathbb{R}^5$, and satisfy the condition (\star) .*

Let $z \in C^2(\Omega)$ be a singular solution to (3) for these coefficients. Then z satisfies the Heinz-Beyerstedt condition, and its limit gradient at the origin is a regular, strictly convex Jordan curve in \mathbb{R}^2 .

To prove this corollary we simply need to go over the proof of Theorem 2 and keep track of where does the analyticity condition play a role. In this way, we have that:

- The proof of Lemma 1 works under the hypothesis that A, \dots, E are of class $C^{1,1}(\mathcal{U})$. So, $z(x, y)$ satisfies the Heinz-Beyerstedt condition.
- The underlying conformal structure of z is that of an annulus, by Lemma 2. That is, the domain $\Lambda \subset \mathbb{C}$ is conformal to some annulus \mathbb{A}_R .
- The bootstrapping method in the proof of Theorem 2 gives that $(x, y, z) \in C^{3,\mu}(\Gamma_r \cup \mathbb{R})$ and $(p, q) \in C^{2,\mu}(\Gamma_r \cup \mathbb{R})$. Furthermore, we can deduce that the limit gradient of z at the origin is a regular strictly convex Jordan curve $\gamma(u) = (p(u), q(u))$, which is now of class $C^{2,\mu}$. This fact is a direct consequence of the condition (26), which still can be deduced from (34) and the Hopf Lemma. Observe that the HeB-condition ensures that the functions (30) are continuous along the real axis and so formula (34) makes sense.

6 Geometric applications

In this section we will explain how our results can be applied to several geometric theories modeled by a Monge-Ampère equation of type (3) which satisfy the condition (\star) .

6.1 Graphs of prescribed curvature in warped product 3-spaces

Recall that the equation for the extrinsic curvature K_{ext} of a graph $z = z(x, y)$ in \mathbb{R}^3 is

$$rt - s^2 = K_{\text{ext}}(1 + p^2 + q^2)^2. \quad (37)$$

In [GHM] Gálvez, Hauswirth and Mira classified the isolated singularities of graphs with $K_{\text{ext}} = 1$ in \mathbb{R}^3 . Their result can be restated as follows (see [GHM, Corollary 13])

Theorem 4 (Galvez-Hauswirth-Mira). *Let \mathcal{A} be the space of graphs $z = z(x, y)$ in \mathbb{R}^3 with $K_{\text{ext}} = 1$ that have a point $p_0 \in \mathbb{R}^3$ as a (non-removable) isolated singularity and extend continuously to this singularity. Then \mathcal{A} is in one-to-one correspondence with the class of real analytic regular convex Jordan curves in \mathbb{S}_+^2 .*

This correspondence associates to each such graph its associated limit gradient at the singularity.

As an application of Theorem 3 we can provide a substantial generalization of Theorem 4. For that, we substitute the ambient space \mathbb{R}^3 by an arbitrary three-dimensional warped product space, and the constant extrinsic curvature by an arbitrary prescribed function, not necessarily constant. Also, we just require the graph to be bounded, and not to extend continuously to the puncture. We explain this next.

Let $\mathcal{D} \times_f \mathbb{R}$ be a three-dimensional warped product space, where $\mathcal{D} \subset \mathbb{R}^2$ is endowed with a conformal metric $g = \lambda(dx^2 + dy^2)$ for some function $\lambda > 0$ (see the Appendix for some background on warped product three-spaces). Then, a computation shows that the extrinsic curvature K_{ext} of an immersed graph $z = z(x, y)$ in $\mathcal{D} \times_f \mathbb{R} \subset \mathbb{R}^3$ is given by equation (3) for the coefficients

$$\begin{aligned} A &= \frac{p\lambda_x}{2\lambda} - \frac{q\lambda_y}{2\lambda} - \frac{q^2 f'}{f} - \frac{f'}{2}\lambda, & B &= \frac{p\lambda_y}{2\lambda} + \frac{q\lambda_x}{2\lambda} + \frac{pqf'}{f}, \\ C &= -\frac{p\lambda_x}{2\lambda} + \frac{q\lambda_y}{2\lambda} - \frac{p^2 f'}{f} - \frac{f'}{2}\lambda, & E &= K_{\text{ext}}(f\lambda + p^2 + q^2)^2 - AC + B^2. \end{aligned} \quad (38)$$

If we substitute in (38) K_{ext} by a smooth function $\mathcal{K}(x, y, z, p, q) > 0$, we get an elliptic Monge-Ampère equation that satisfies condition (\star) .

Observe that, by Theorem 7, any graph in $\mathcal{D}^2 \times_f \mathbb{R}$ with $K_{\text{ext}} > 0$ which is bounded around an isolated singularity automatically extends continuously across the singularity. We will hence assume this continuous extension property in the next theorem.

Theorem 5. *Let $\mathcal{D} \times_f \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R}$ be a real analytic warped product three-manifold, let $p_0 = (q_0, z_0) \in \mathcal{D} \times_f \mathbb{R}$, and let \mathcal{O} be an open set of the tangent bundle $T(\mathcal{D} \times_f \mathbb{R}) \equiv \mathcal{D} \times \mathbb{R} \times \mathbb{R}^3$, such that $\mathcal{O} \cap (\{p_0\} \times \mathbb{R}^3) \neq \emptyset$.*

Let $\mathcal{K} : \mathcal{O} \rightarrow \mathbb{R}$ be a positive, real analytic function, and consider the moduli space \mathcal{A}_1 of graphs Σ in $\mathcal{D} \times_f \mathbb{R}$ over some punctured disk around $q_0 \in \mathcal{D}$ such that:

1. Σ has at $q_0 \in \mathcal{D}$ an isolated singularity, and is bounded around q_0 , with $p_0 \in \overline{\Sigma}$.

2. The extrinsic curvature of Σ is given by the function \mathcal{K} ; that is, if we view Σ as a graph $z = z(x, y)$ over a punctured disk around $q_0 \in \mathcal{D}$, the extrinsic curvature K_{ext} of Σ at the point (x, y, z) is $\mathcal{K}(x, y, z, z_x, z_y)$.

Then \mathcal{A}_1 is in one-to-one correspondence with the space \mathcal{A}_2 of all the regular, real analytic, convex Jordan curves contained in the non-empty planar open set $\mathcal{O} \cap (\{p_0\} \times \mathbb{R}^2) \subset \mathbb{R}^2$.

Proof. To be coherent with our previous notation, we will suppose that $p_0 = (0, 0, 0)$ in the (x, y, z) coordinates of the warped product. As Σ is bounded around the isolated singularity and has $K_{\text{ext}} > 0$, Theorem 7 in the Appendix implies that the graph extends continuously to the puncture with $z(q_0) = p_0$, and is uniformly non-vertical. Then, by the computation (38) we see that the graphs $\Sigma \in \mathcal{A}_1$ are exactly the singular solutions to the Monge-Ampère equation (3)-(38). Therefore, Theorem 5 follows from Theorem 3. \square

Remark 3. In the case where the ambient space is a Riemannian product $M^2 \times \mathbb{R}$ (i.e. $f = 1$), the hypothesis that Σ is bounded around the singularity is superfluous, by Theorem 8. This includes the case where the ambient space is \mathbb{R}^3 .

Theorem 5 can be applied to the case in which the ambient space is a three-dimensional space of constant curvature endowed with a warped product structure. For example (see e.g. [GJM]):

1. The Euclidean space \mathbb{R}^3 minus one point can be seen as a non-trivial warped product by considering spherical coordinates:

$$\mathbb{R}_*^3 \equiv \mathbb{R}^3 \setminus \{0\} \equiv (\mathbb{S}^2 \times \mathbb{R}^+, z^2 g_{\mathbb{S}^2} + dz^2).$$

In this model the equation $z = z(x, y)$ provides graphs over spheres.

2. The half-space model of the hyperbolic 3-space $\mathbb{H}^3 = (\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}, \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2))$ can be written as the warped product

$$(\mathbb{R}^2 \times \mathbb{R}, e^{-2z}(dx^2 + dy^2) + dz^2). \quad (39)$$

In this model, the graphs $z = z(x, y)$ correspond geometrically to graphs in \mathbb{H}^3 over horospheres.

3. Using the warped representation of \mathbb{R}_*^3 mentioned above we can pass from the Poincaré model of \mathbb{H}^3 , $(\mathbb{B}^3, \frac{4}{(1-|p|^2)^2} g_{\mathbb{R}^3})$, to the model

$$\mathbb{H}_*^3 = \mathbb{H}^3 \setminus \{0\} \equiv (\mathbb{S}^2 \times \mathbb{R}^+, \sinh^2(z) g_{\mathbb{S}^2} + dz^2).$$

As before, $z = z(x, y)$ gives rise to graphs over totally umbilical spheres in \mathbb{H}^3 .

4. In the case of \mathbb{S}^3 , we consider the warped model

$$\mathbb{S}^3 \setminus \{\text{north, south}\} \equiv (\mathbb{S}^2 \times \mathbb{R}, \sin^2(z)g_{\mathbb{S}^2} + dz^2).$$

Again, $z = z(x, y)$ describes geometrically graphs over a totally umbilical sphere.

Observe that Theorem 5 provides the classification of isolated singularities of bounded graphs of prescribed analytic positive extrinsic curvature in all these models. We omit the specific statement of the results, as the general case was already explained in Theorem 5.

The next corollary is a consequence of Theorem 5 and Corollary 4 in the Appendix.

Corollary 2. *Let us consider the warped product model for \mathbb{H}^3 given by (39), i.e. $\mathbb{H}^3 \equiv \mathbb{R}^2 \times_{e^{-2z}} \mathbb{R}$. Let $\mathcal{K}(x, y)$ be a real analytic positive function in some disk $D \subset \mathbb{R}^2$ containing $p_0 = (0, 0)$, and let Σ be a real analytic graph $z = z(x, y)$ on the punctured disk $D^* = D \setminus \{p_0\}$. Then, exactly one of the following three situations happens:*

1. Σ extends analytically across p_0 .
2. Σ extends continuously (but not C^1 -smoothly) across p_0 , and its limit gradient at p_0 is an analytic, regular, strictly convex Jordan curve γ . In that case Σ can be recovered in terms of γ by the process explained in Theorem 1.
3. The height function of Σ over \mathcal{H} tends to $+\infty$ or to $-\infty$ at p_0 . In particular, the metric of Σ is complete around the puncture. This third case cannot happen if $\mathcal{K}(p_0) > 1$.

Let us note that, by the Gauss equation $K_G = K_{\text{ext}} + \varepsilon$, prescribing the extrinsic curvature K_{ext} on a graph in a space of constant curvature ε is equivalent to prescribing the (intrinsic) Gaussian curvature K_G of the graph.

6.2 Embedded isolated singularities of surfaces with positive curvature in \mathbb{R}^3

Let $\psi : \Omega \rightarrow \mathbb{R}^3$ be an immersion of the punctured disc Ω into \mathbb{R}^3 and assume that ψ extends continuously but not C^1 -smoothly to the origin. Following [GaMi], we say in these conditions that ψ has an *embedded isolated singularity* at $p = \psi(0) \in \mathbb{R}^3$ if there is a punctured neighborhood $U \subset \Omega$ of the origin such that $\psi(U)$ is an embedded surface.

By Theorem 13 in [GaMi], it follows that if a surface of positive curvature in \mathbb{R}^3 has an embedded isolated singularity at some $p \in \mathbb{R}^3$, then the surface can be viewed around this point as a convex graph over a punctured disc in some direction of \mathbb{R}^3 .

Besides, it is easy to observe that for any direction $v_0 \in \mathbb{S}^2$ a curve $\beta(u)$ in the hemisphere $\mathbb{S}^2 \cap \{x \in \mathbb{R}^3 : \langle x, v_0 \rangle > 0\}$ is regular and strictly convex if and only if so is the planar curve $\gamma(u)$ contained in the plane $\{v_0\}^\perp \subset \mathbb{R}^3$ given by

$$\gamma(u) = \frac{\langle \beta(u), e_1 \rangle}{\langle \beta(u), v_0 \rangle} e_1 + \frac{\langle \beta(u), e_2 \rangle}{\langle \beta(u), v_0 \rangle} e_2,$$

where $\{e_1, e_2, v_0\}$ is a positively oriented orthonormal basis of \mathbb{R}^3 .

With all of this, and recalling that the equation for the extrinsic curvature of a graph $z = z(x, y)$ in \mathbb{R}^3 is given by (37), and invariant by isometries of \mathbb{R}^3 , we obtain the following theorem as a corollary of Theorem 3.

Theorem 6. *Let $\mathcal{K} : \mathcal{O} \subset \mathbb{R}^3 \rightarrow (0, \infty)$ be a positive real analytic function defined on an open set $\mathcal{O} \subset \mathbb{R}^3$ containing a given point $p \in \mathbb{R}^3$. Let \mathcal{A}_1 denote the class of all the surfaces Σ in \mathbb{R}^3 that have p as an embedded isolated singularity, and whose extrinsic curvature at every point $(x, y, z) \in \Sigma \cap \mathcal{O}$ is given by $\mathcal{K}(x, y, z)$.*

Then, the map that sends each surface in \mathcal{A}_1 to its limit unit normal at the singularity[†] provides a one-to-one correspondence between \mathcal{A}_1 and the class \mathcal{A}_2 of regular, analytic, strictly convex Jordan curves in \mathbb{S}^2 .

Theorem 6 generalizes [GHM, Corollary 13], which covers the case $\mathcal{K} = \text{const}$.

6.3 Spacelike surfaces with $K_{\text{ext}} > 0$ in \mathbb{L}^3

The Monge-Ampère equation (3), and more specifically (8), also appears when dealing with spacelike graphs of prescribed negative Gaussian curvature in the 3-dimensional Lorentz-Minkowski space \mathbb{L}^3 . Let us recall that \mathbb{L}^3 can be seen as \mathbb{R}^3 endowed with the Lorentzian flat metric $\langle \cdot, \cdot \rangle = dx^2 + dy^2 - dz^2$, and that a graph $z = z(x, y)$ in \mathbb{L}^3 is called *spacelike* if the metric that it inherits from the one of \mathbb{L}^3 is Riemannian. Alternatively, the graph $z = z(x, y)$ is spacelike if $z_x^2 + z_y^2 < 1$ everywhere.

If we denote by K_G the Gaussian curvature of a spacelike surface in \mathbb{L}^3 , its extrinsic curvature is given by $K_{\text{ext}} = -K_G$. In this way, a computation shows that the extrinsic curvature of a spacelike graph $z = z(x, y)$ in \mathbb{L}^3 is given by

$$rt - s^2 = K_{\text{ext}}(1 - p^2 - q^2)^2, \quad p^2 + q^2 < 1. \quad (40)$$

If we substitute in (40) K_{ext} by a function $\mathcal{K}(x, y, z, p, q) > 0$, we obtain an elliptic Monge-Ampère equation for which condition (\star) is trivially satisfied.

There is an interesting feature of this equation. Let $z(x, y)$ be a graph in \mathbb{L}^3 with $K_{\text{ext}} > 0$ over the punctured disk Ω , which satisfies the ellipticity condition $z_x^2 + z_y^2 < 1$. Also, assume that z does not extend C^1 to the origin. Then, for every $\varepsilon > 0$ such that $\partial B(0, \varepsilon) \subset \Omega$, the maximum c_0 of $z_x^2 + z_y^2$ in $\partial B(p_0, \varepsilon)$ satisfies $c_0 < 1$. Now, as by (40) the graph $z = z(x, y)$ is locally convex, we deduce by convexity that $z_x^2 + z_y^2 \leq c_0 < 1$ in $\overline{B(0, \varepsilon)} \setminus \{0\} \subset \Omega$. In particular, the functions z_x, z_y are bounded near the origin and z extends continuously to $(0, 0)$.

In other words, when we view (40) as a Monge-Ampère equation for $K_{\text{ext}} = \mathcal{K}(x, y, z, p, q) > 0$, any solution on Ω which does not extend C^1 -smoothly to the origin is automatically a singular solution of the equation, in the sense of Definition 1.

With this, Theorem 3 provides the following consequence.

[†]That is, the set of vectors $v_0 \in \mathbb{S}^2$ for which there exist points $p_n \in \Sigma$ converging to p such that the corresponding unit normals $N(p_n) \in \mathbb{S}^2$ of Σ at those points converge to v_0 .

Corollary 3. *Let $\mathcal{K}(x, y, z, p, q) > 0$ be a positive real analytic function on an open set $\mathcal{U} \subset \mathbb{R}^5$ such that the set $\mathcal{H} := \{(p, q) \in \mathbb{R}^2 : (0, 0, 0, p, q) \in \mathcal{U}\}$ is non-empty. Let \mathcal{A}_1 denote the class of spacelike graphs $z = z(x, y)$ in \mathbb{L}^3 over some punctured disk $\Omega \subset \mathbb{R}^2$ such that:*

1. $z(x, y)$ extends continuously but not C^1 -smoothly to the origin, with $z(0, 0) = 0$.
2. The extrinsic curvature of the graph is given by $K_{\text{ext}}(x, y) = \mathcal{K}(x, y, z, z_x, z_y)$ for every $(x, y) \in \Omega$.

Then the map sending each element of \mathcal{A}_1 to its limit gradient at the origin is a bijective correspondence between \mathcal{A}_1 and the class \mathcal{A}_2 of regular, real analytic, strictly convex Jordan curves $\gamma \subset \mathcal{H} \cap \mathbb{D} \subset \mathbb{R}^2$, where \mathbb{D} is the unit disc.

6.4 Elliptic Weingarten graphs

An orientable surface in \mathbb{R}^3 is called a *linear Weingarten* surface if there exist three real numbers a, b, c not all zero, such that

$$2aH + bK = c, \quad (41)$$

where K and H are respectively the Gauss curvature and the mean curvature of the surface. The surface is called an *elliptic* linear Weingarten surface when

$$a^2 + bc > 0. \quad (42)$$

Observe that in this elliptic case we can suppose up to a change of sign in (41) that $c \geq 0$. This family contains the class of surfaces of constant mean curvature ($b = 0$) and surfaces with positive constant Gauss curvature ($a = 0$). For a graph $z = z(x, y)$ in \mathbb{R}^3 the condition of being an elliptic linear Weingarten surface is given by the following equation:

$$rt - s^2 + \frac{a}{b}\sqrt{1 + p^2 + q^2}((1 + p^2)t - 2pqs + (1 + q^2)r) = \frac{c}{b}(1 + p^2 + q^2)^2. \quad (43)$$

Equation (43) is an elliptic Monge-Ampère equation that can be written in the form (3) for

$$\begin{aligned} A &= \frac{a}{b}\sqrt{1 + p^2 + q^2}(1 + q^2), & B &= \frac{a}{b}\sqrt{1 + p^2 + q^2}(-pq) \\ C &= \frac{a}{b}\sqrt{1 + p^2 + q^2}(1 + p^2), & E &= \frac{c}{b}(1 + p^2 + q^2)^2. \end{aligned} \quad (44)$$

These coefficients do not satisfy the condition (\star) , so Theorem 3 does not provide a direct classification result as in previous subsections. Still, by the procedure explained in Section 4 we can give a way to construct linear Weingarten graphs with the underlying conformal structure of an annulus in terms of regular analytic strictly convex Jordan curves. However, we must emphasize that there are elliptic linear Weingarten surfaces with isolated singularities that are not recovered by this process, see Figure 5.1.

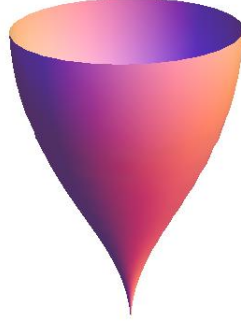


Figure 2: An isolated singularity of a rotational elliptic linear Weingarten graph, whose gradient tends to infinity at the puncture.

Appendix: Isolated singularities of graphs in warped products

Given a Riemannian surface (M^2, g) and a smooth function $f : \mathbb{R} \rightarrow (0, \infty)$, we define the *three-dimensional warped product* $M^2 \times_f \mathbb{R}$ as the Riemannian manifold $(M^2 \times \mathbb{R}, \langle, \rangle)$, where

$$\langle, \rangle = f(t)g + dt^2.$$

A surface Σ in $M^2 \times_f \mathbb{R}$ is a *graph* if $\pi : \Sigma \rightarrow \pi(\Sigma)$ is a diffeomorphism, where π stands for the projection $M^2 \times_f \mathbb{R} \rightarrow M^2$. If we choose coordinates (x, y) on a domain $\mathcal{D} \subset M^2$ containing $\pi(\Sigma)$, then the graph Σ in the coordinates (x, y, t) is given by $z = z(x, y)$, where $z(x, y)$ is a smooth function. We will call z the *height function* of the graph.

Definition 3. Let Σ be a graph in $M^2 \times_f \mathbb{R}$ over a punctured disk $D^* \subset M^2$ around some $p_0 \in M^2$. If Σ does not extend as a C^1 graph to $D = D^* \cup \{p_0\}$, we will call p_0 an isolated singularity of Σ .

Given a graph $\Sigma \subset M^2 \times_f \mathbb{R}$, we can orient it so that its unit normal N satisfies $\langle N, \partial_t \rangle \in (0, 1]$, where ∂_t is the derivative of the vertical coordinate in $M^2 \times_f \mathbb{R}$. We will call $\nu := \langle N, \partial_t \rangle$ the *angle function* associated to Σ , and we will say that Σ is *uniformly non-vertical* if $\nu \geq c > 0$ in Σ .

In what follows we will denote by K_{ext} the *extrinsic curvature function* of the oriented graph Σ , i.e. the determinant of the second fundamental form II of Σ with respect to its first fundamental form. Then, the condition $K_{\text{ext}} > 0$ is equivalent to the property that II is (positive or negative) definite at every point.

In this Appendix we will prove the following three results on the geometry of isolated singularities for graphs of positive extrinsic curvature in warped product three-manifolds.

Theorem 7. *Let Σ be a graph in $M^2 \times_f \mathbb{R}$ with $K_{\text{ext}} > 0$. Assume that Σ has an isolated singularity at $p_0 \in M^2$ and is bounded around p_0 . Then Σ extends across p_0 as a continuous graph and is uniformly non-vertical.*

Theorem 8. *Let Σ be a graph with $K_{\text{ext}} > 0$ in the Riemannian product space $M^2 \times \mathbb{R}$ (i.e. $f = 1$). Assume that Σ has an isolated singularity at $p_0 \in M^2$. Then Σ extends across p_0 as a continuous graph and is uniformly non-vertical.*

Theorem 9. *Let Σ be a graph with $K_{\text{ext}} > 0$ in a warped product space $M^2 \times_f \mathbb{R}$ which is a Hadamard manifold. Assume that Σ has an isolated singularity at $p_0 \in M^2$ and that Σ is not bounded around p_0 . Then, if z denotes the height function of Σ ,*

$$\lim_{p \rightarrow p_0} z(p) = \pm\infty, \quad p \in M^2.$$

Remark 2. *Theorem is also true if, instead of assuming that $M^2 \times_f \mathbb{R}$ is a Hadamard manifold, we ask for the following property (P) to hold for some strongly convex geodesic disk $D_r \subset M^2$ of radius r centered at p_0 .*

Property (P): *For any two points in $D_r \times \mathbb{R} \subset M^2 \times_f \mathbb{R}$ there is a unique minimizing geodesic joining both points, and moreover, this geodesic is totally contained in $D_r \times \mathbb{R}$.*

Observe that any Hadamard manifold $M^2 \times_f \mathbb{R}$ verifies property (P). Basic examples of this type of manifolds are \mathbb{R}^3 , \mathbb{H}^3 and $\mathbb{H}^2 \times \mathbb{R}$.

We begin with the proof of these three results.

Let (M^2, g) be an oriented Riemannian surface and $p_0 \in M^2$. For a fixed unit vector $\xi \in T_{p_0}M$, let $\gamma(v)$ be the unique geodesic in M^2 with initial conditions $\gamma(0) = p_0$, $\gamma'(0) = \xi$. Let \exp and J denote, respectively, the exponential map and the complex structure of (M^2, g) . Then, for $\varepsilon > 0$ small enough, the map

$$(u, v) \longmapsto \exp_{\gamma(v)}(uJ\gamma'(v))$$

defines a diffeomorphism from $R_\varepsilon := (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$ into a neighbourhood $U \subset M^2$ of p_0 , such that the metric g is expressed with respect to (u, v) as

$$g = du^2 + G(u, v)dv^2$$

for a positive smooth function $G(u, v)$ in R_ε with $G(0, v) = 1$ for all $v \in (-\varepsilon, \varepsilon)$.

Observe that in these coordinates, each curve $v = \text{const.}$ in R_ε corresponds to a geodesic of (M^2, g) .

Let now Σ be a graph in $M^2 \times_f \mathbb{R}$ with an isolated singularity at p_0 . If we parametrize $U \times \mathbb{R}$ in terms of the (u, v, t) coordinates defined above, then Σ is written in a neighborhood of p_0 as

$$\Sigma = \{(u, v, z(u, v)) : (u, v) \in \Omega \subset R_\varepsilon \setminus \{(0, 0)\}\}$$

for some smooth function z defined on a punctured disk Ω centered at the origin.

Let $\bar{\partial}_u, \bar{\partial}_v$ denote the partial derivatives in Σ with respect to (u, v) , i.e.

$$\bar{\partial}_u = \partial_u + z_u \partial_t, \quad \bar{\partial}_v = \partial_v + z_v \partial_t,$$

and let η be the (non-unit) upwards pointing normal vector field to Σ

$$\eta := -z_u \partial_u - \frac{z_v}{G} \partial_v + f \partial_t.$$

Then, bearing in mind that the Levi-Civita connection ∇ in $M^2 \times_f \mathbb{R}$ in the coordinates (u, v, t) satisfies

$$\nabla_{\partial_u} \partial_u = -\frac{f(t)}{2} \partial_t, \quad \nabla_{\partial_u} \partial_t = \frac{f'(t)}{2f(t)} \partial_u, \quad \nabla_{\partial_t} \partial_t = 0,$$

a simple computation shows that the second fundamental form II of Σ verifies

$$\begin{aligned} II(\bar{\partial}_u, \bar{\partial}_u) &= \langle \nabla_{\bar{\partial}_u} \bar{\partial}_u, \frac{\eta}{\|\eta\|} \rangle \\ &= \frac{1}{\|\eta\|} \left(-f'(z) z_u^2 + f(z) \left(z_{uu} - \frac{f'(z)}{2} \right) \right). \end{aligned}$$

If we assume now that $K_{\text{ext}} > 0$ for Σ , then

$$-f'(z) z_u^2 + f(z) \left(z_{uu} - \frac{f'(z)}{2} \right) \neq 0 \quad (45)$$

for every $(u, v) \in \Omega$. Alternatively, we can rewrite (45) as

$$\frac{\partial}{\partial u} \left(\frac{z_u}{f'(z)} \right) > \frac{f'(z)}{2f(z)} \quad \text{or else} \quad \frac{\partial}{\partial u} \left(\frac{z_u}{f'(z)} \right) < \frac{f'(z)}{2f(z)}. \quad (46)$$

Proof of Theorem 7: We assume, for instance, that the first inequality in (46) holds (the argument is similar with the second inequality). As z is bounded by hypothesis, there is some $c_0 \in \mathbb{R}$ such that, for $(u, v) \in \Omega \subset \mathbb{R}_\varepsilon \setminus \{(0, 0)\}$,

$$\frac{\partial}{\partial u} \left(\frac{z_u}{f'(z)} \right) > \frac{f'(z)}{2f(z)} \geq c_0,$$

and therefore

$$\frac{\partial}{\partial u} \left(\frac{z_u}{f'(z)} - c_0 u \right) > 0.$$

This condition easily implies that

$$\frac{z_u}{f'(z)} - c_0 u$$

is bounded in Ω , from where z_u is also bounded in Ω .

Recall now that, by construction, the coordinates (u, v) depend on the arbitrary unit vector ξ which determines the ∂_v direction. So, a different choice $\widehat{\xi}$ of the vector ξ will result in new coordinates $(\widehat{u}, \widehat{v})$ for which $z_{\widehat{u}}$ will be bounded. By choosing $\widehat{\xi}$ so that $\{\partial_u, \partial_{\widehat{u}}\}$ are linearly independent at $(0, 0)$, it is easy to deduce then that z_v is bounded around $(0, 0)$.

On the other hand, a computation shows that the angle function ν of Σ satisfies

$$\nu^2 = \langle N, \partial_t \rangle^2 = \frac{1}{\|\eta\|^2} \langle \eta, \partial_t \rangle^2 = \frac{f(z)}{f(z) + z_u^2 + \frac{z_v^2}{G}}. \quad (47)$$

As z, z_u and z_v are bounded around $(0, 0)$, we conclude from (47) that $\nu^2 \geq c > 0$ around $(0, 0)$, i.e. Σ is uniformly non-vertical around the isolated singularity. Finally, that z is continuous at $(0, 0)$ follows from the boundedness of z_u and z_v . This completes the proof of Theorem 7. \square

Proof of Theorem 8: By the condition $f = 1$, we get from (45) that z_{uu} has a constant sign on the punctured disk $\Omega \subset \mathbb{R}_\varepsilon \setminus \{(0, 0)\}$. Thus z_u is bounded. The rest of the argument is identical to the one used in the proof of Theorem 7. \square

Proof of Theorem 9: Let $D_r \subset M^2$ be a strongly convex geodesic disk of radius r centered at p_0 , and assume that property **(P)** holds in $D_r \times \mathbb{R} \subset M^2 \times_f \mathbb{R}$ (this happens for instance if $M^2 \times_f \mathbb{R}$ is a Hadamard manifold). We will assume without loss of generality that the graph Σ is well defined on $D_r \setminus \{p_0\}$, and the second fundamental form of Σ is positive definite for the upwards pointing unit normal N of Σ (recall that $K_{\text{ext}} > 0$ on Σ). This implies, using that the vertical planes $\gamma \times \mathbb{R}$ over a geodesic $\gamma \subset M^2$ are totally geodesic surfaces in $M^2 \times_f \mathbb{R}$, that any geodesic in $M^2 \times_f \mathbb{R}$ which is tangent to Σ at some point $p \in \Sigma$ lies below Σ around p .

We define the *epigraph* of z by

$$\text{epi}(z) = \{(p, t) \in D_r \times \mathbb{R} : t \geq z(p), p \neq p_0\}.$$

Claim 1: $\overline{\text{epi}(z)}$ is a convex subset of $\overline{D_r} \times \mathbb{R}$.

To prove this claim, we take two points $(p_1, t_1), (p_2, t_2)$ in $\overline{\text{epi}(z)} \subset \overline{D_r} \times \mathbb{R}$ and prove that the unique geodesic Γ in $\overline{D_r} \times \mathbb{R}$ joining them is contained in $\overline{\text{epi}(z)}$. We distinguish several cases.

Case 1: the geodesic γ joining p_1 and p_2 in $\overline{D_r}$ does not pass through p_0 .

First note that if $p_1 = p_2$, the geodesic Γ corresponds to a vertical segment, so the property holds.

Consider the totally geodesic plane over γ , that is, $\gamma \times \mathbb{R} \subset D_r \times \mathbb{R}$. Then, the geodesic Γ is contained in $\gamma \times \mathbb{R}$. Let $\alpha = (\gamma \times \mathbb{R}) \cap \Sigma$.

Let θ_0 be the angle that the geodesic $\Gamma \subset \gamma \times \mathbb{R}$ makes with the vertical direction ∂_t at the point (p_1, t_1) , and consider the family $\{\Gamma_\theta\}_{\theta \in [0, \theta_0]}$ of geodesics in $\gamma \times \mathbb{R}$ starting at (p_1, t_1) and making an angle θ with ∂_t at this initial point. Note that Γ_0 is $\{p_1\} \times [t_1, \infty)$, which does not intersect α , that $\Gamma_{\theta_0} = \Gamma$, and that all such geodesics only intersect at the initial point (p_1, t_1) .

Once here, observe that the existence of a point $q \in \Gamma$ not lying in $\overline{\text{epi}(z)}$ would mean that α is *above* Γ around q in $\gamma \times \mathbb{R}$. But that would mean that some of the geodesics Γ_θ lies above α in $\gamma \times \mathbb{R}$ and touches α tangentially at some point. This is a contradiction with the fact that Σ has positive definite second fundamental form for the upwards pointing unit normal, and hence lies locally above all its tangent geodesics. This completes the proof of the convexity of $\overline{\text{epi}(z)}$ in Case 1.

Case 2: the geodesic γ joining p_1 and p_2 in $\overline{D_r}$ passes through p_0 .

Let $\{x_n\}$ be a sequence of points in D_r with $x_n \rightarrow p_2$, and such that the geodesic in D_r joining x_n and p_1 does not pass through p_0 . Then, we can also take $(t_n)_n$ such that $(x_n, t_n) \in \overline{\text{epi}(z)}$ and $(x_n, t_n) \rightarrow (p_2, t_2)$ as $n \rightarrow \infty$. By Case 1, the geodesic Γ_n joining (p_1, t_1) with (x_n, t_n) is contained in $\overline{\text{epi}(z)}$. Taking limits, Γ_n converge to the geodesic Γ joining (p_1, t_1) with (p_2, t_2) . In particular, Γ is contained in $\overline{\text{epi}(z)}$ as we wanted to prove. We remark that this argument also holds in the case $p_0 = p_2$.

Case 3: $p_1 = p_2 = p_0$.

In this case we take two points (p_0, t_1) and (p_0, t_2) in $\overline{\text{epi}(z)}$ with $t_1 < t_2$. Take a sequence $(x_n, t_n) \rightarrow (p_0, t_1)$. Then, the vertical segments

$$\Gamma_n := \{x_n\} \times [t_n, t_n + t_2 - t_1] \subset \overline{\text{epi}(z)}$$

are geodesics, so by taking limits $\Gamma_n \rightarrow \Gamma = \{p_0\} \times [t_1, t_2]$, which is also contained in $\overline{\text{epi}(z)}$.

Thus, we have proved that $\overline{\text{epi}(z)}$ is a convex set of $\overline{D_r} \times \mathbb{R}$.

Let us also observe that there is some $t_0 \in \mathbb{R}$ such that $(p_0, t_0) \in \text{int}(\overline{\text{epi}(z)})$. For this, let $p_1, p_2 \in D_r$ such that p_0 lies in the geodesic γ joining p_1 with p_2 . If we take $t_1 > z(p_1)$, then $(p_1, t_1) \in \text{int}(\overline{\text{epi}(z)})$ and the geodesic in $D_r \times \mathbb{R}$ joining (p_1, t_1) with $(p_2, z(p_2))$ passes through some point of the form (p_0, t_0) . By standard convexity arguments, $(p_0, t_0) \in \text{int}(\overline{\text{epi}(z)})$.

Consider now $(p_0, t_0) \in \text{int}(\overline{\text{epi}(z)})$, and let $\varepsilon > 0$ so that $D_\varepsilon \times (t_0 - \varepsilon, t_0 + \varepsilon) \subset \overline{\text{epi}(z)}$, where $D_\varepsilon \subset D_r$ is a geodesic disk of radius ε centered at p_0 . Then, $D_\varepsilon \times (t_0 - \varepsilon, \mathbb{R}) \subset \overline{\text{epi}(z)}$ from where it follows that $(p_0, t_1) \in \text{int}(\overline{\text{epi}(z)})$ for all $t_1 > t_0$. By convexity, there are two possibilities:

1. $\overline{\text{epi}(z)} \cap (\{p_0\} \times \mathbb{R}) = \{p_0\} \times \mathbb{R}$, with $\{p_0\} \times \mathbb{R} \subset \text{int}(\overline{\text{epi}(z)})$, or
2. $\overline{\text{epi}(z)} \cap (\{p_0\} \times \mathbb{R}) = \{p_0\} \times [h, \infty)$ with $\{p_0\} \times (0, \infty) \subset \text{int}(\overline{\text{epi}(z)})$

We will prove next that in the second case we have $\lim_{p \rightarrow p_0} z(p) = h$. An analogous argument would prove that $\lim_{p \rightarrow p_0} z(p) = -\infty$ in the first case. Observe that this would conclude the proof of Theorem 9, as the second case is impossible since we are assuming that z is not bounded around p_0 .

Let $\{p_n\}$ be a sequence of points in $D_r \setminus \{p_0\}$ converging to p_0 . As we are in the second case above, there exist some $\varepsilon > 0$ and $\delta > 0$ arbitrarily small such that

$$(p_0, h + \delta) \in D_\varepsilon \times (h + \delta - \varepsilon, h + \delta + \varepsilon) \subset \overline{\text{epi}(z)}$$

and

$$(p_0, h - \delta) \in D_\varepsilon \times (h - \delta - \varepsilon, h - \delta + \varepsilon) \subset (D_r \times \mathbb{R}) \setminus \overline{\text{epi}(z)}.$$

In particular, we have

$$D_\varepsilon \times [h + \delta, \infty) \subset \text{int}(\overline{\text{epi}(z)}) \quad \text{and} \quad D_\varepsilon \times (-\infty, h - \delta] \subset (D_r \times \mathbb{R}) \setminus \text{int}(\overline{\text{epi}(z)}).$$

Now, observe that, since $(p_n, z(p_n))$ lies in the boundary of $\text{epi}(z)$, it holds $(p_n, z(p_n)) \notin \text{int}(\overline{\text{epi}(z)}) \cup \left((D_r \times \mathbb{R}) \setminus \text{int}(\overline{\text{epi}(z)}) \right)$. Thus, choosing n_0 so that if $n \geq n_0$ we have $p_n \in D_\varepsilon$, we can conclude from the previous condition that $|z(p_n) - h| < \delta$ for every $n \geq n_0$.

This proves the claim and completes the proof of Theorem 9. \square

Corollary 4. *Let $\Sigma \subset M^2 \times_f \mathbb{R}$ be a graph with $K_{\text{ext}} > 0$ over a punctured disk $D^* \subset M^2$ centered at $p_0 \in M^2$, and assume that $M^2 \times_f \mathbb{R}$ is a Hadamard manifold. Then, exactly one of the following three situations happen:*

1. Σ extends as a smooth C^1 -graph across p_0 .
2. Σ extends continuously (but not C^1 -smoothly) across p_0 , and is uniformly non-vertical at p_0 .
3. The height function of Σ tends to $+\infty$ or to $-\infty$ at p_0 . In particular, the metric of Σ is complete around the puncture.

Moreover, assume that the sectional curvature of $M^2 \times_f \mathbb{R}$ is bounded from below on a tubular neighbourhood of the geodesic $\{p_0\} \times \mathbb{R}$ by a number $i(p_0) \in \mathbb{R}$. Then the third situation above cannot happen provided $K_{\text{ext}} > -i(p_0)$.

Proof. By Theorems 7 and 9 we only need to prove the final assertion. This is a consequence of the Gauss equation for Σ in $M^2 \times_f \mathbb{R}$, which implies that if $K_{\text{ext}} > -i(p_0)$, the Gaussian curvature of Σ is bounded from below by a positive constant around p_0 .

But now, observe that in $M^2 \times_f \mathbb{R}$, the distance between (p_1, t_1) and (p_2, t_2) is at least $|t_2 - t_1|$. Hence, the fact that the height function of Σ tends to $\pm\infty$ around p_0 indicates that there exist points $p, q \in \Sigma$ arbitrarily far away from each other. This contradicts that on a surface with curvature bounded from below by some $c > 0$, the length of any minimizing geodesic is $\leq \pi/\sqrt{c}$. \square

References

- [ACG] J.A. Aledo, R. M. B. Chaves, J. A. Gálvez, The Cauchy Problem for Improper Affine Spheres and the Hessian One Equation, Trans. Amer. Math. Soc. 359 4183-4208 (2007).
- [AlCu] S. Alexander, R. Currier, Non-negatively curved hypersurfaces of hyperbolic space and subharmonic functions, J. London Math. Soc. 41 347-360 (1990).

- [Bey1] R. Beyerstedt, Removable singularities of solutions to elliptic Monge-Ampère equations, *Math. Z.* 208, 363-373 (1991).
- [Bey2] R. Beyerstedt, The behaviour of solutions to elliptic Monge-Ampère equations at singular points. *Math. Z.* 216, 243-256 (1994).
- [Fried] A. Freidman, A new proof and generalizations of the Cauchy-Kowalewski Theorem, *Transactions of the A. M. S.* Vol. 98, No. 1, Jan. (1961).
- [GaMi] J. A. Gálvez, P. Mira, Embedded isolated singularities of flat surfaces in hyperbolic 3-space, *Calc. Var. Partial Diff. Equations* 24 239-260 (2005).
- [GHM] J. A. Gálvez, L. Hauswirth, P. Mira, Surfaces of constant curvature in \mathbb{R}^3 with singularities, preprint.
- [GMM] J. A. Gálvez, A. Martínez, P. Mira, The space of solutions to the Hessian one equation in the finitely punctured plane, *J. Math Pures Appl.* 84, 1744-1757 (2005).
- [GJM] J. A. Gálvez, A. Jiménez, P. Mira, A correspondence for isometric immersions into product spaces and its applications, *J. of Geometry and Physics* 60, 1819-1832 (2010).
- [GiTr] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, (Classics in mathematics) Springer (2001).
- [He] E. Heinz, Über das Randverhalten quasilinearer elliptischer Systeme mit isothermen Parametern. *Math. Z.* 113, 99-105 (1970).
- [HeB] E. Heinz, R. Beyerstedt, Isolated singularities of Monge-Ampère equations, *Calc. Var.* 2, 241-247 (1994).
- [Jor1] K. Jörgens, Über die Lösungen der Differentialgleichung $rt - s^2 = 1$. *Math. Ann.* 127, 330-344 (1954).
- [Jor2] K. Jörgens, Harmonische Abbildungen und die Differentialgleichung $rt - s^2 = 1$. *Math. Ann.* 129, 130-134 (1954).
- [LSZ] A. M. Li, U. Simon, G. S. Zhao, *Global affine differential geometry of hypersurfaces*. De Gruyter Expositions in Mathematics. Walter de Gruyter (1993).
- [Mu1] F. Müller, On the continuation of solutions for elliptic equations in two variables. *Ann. I. H. Poincaré.* 19, 754-776 (2002).
- [Mu2] F. Müller, Analyticity of solutions for semilinear elliptic systems of second order, *Calc. Var.* 15, 257-288 (2002).
- [Ol] V. I. Oliker, Hypersurfaces in \mathbb{R}^{n+1} with prescribed Gaussian curvature and related equations of Monge-Ampère type. *Comm. Partial Differential Equations* 9 807-838 (1984).

[Ve] I. N. Vekua, Generalized Analytic Functions. Oxford London New York Paris,
Pergamon (1962).